## 1. 布朗运动的期望

**Teorema 1**

设 $B_t$ 是布朗运动，则证明

$$E[B_t^{2n}] = (2n-1)!!t^n$$

proof. 由 $B_t$ 是布朗运动，则有：

$$\begin{aligned}
E[B_t^{2n}] &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} x^{2n} e^{-\frac{x^2}{2t}} dx \\
&= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} -t x^{2n-1} d e^{-\frac{x^2}{2t}} \\
&= -\frac{t}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} x^{2n-1} d e^{-\frac{x^2}{2t}} \\
&= -\frac{t}{\sqrt{2\pi t}} (x^{2n-1} e^{-\frac{x^2}{2t}} |_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2t}} dx^{2n-1}) \\
&= -\frac{t}{\sqrt{2\pi t}} \left(0 - (2n-1) \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2t}} x^{2n-2} dx\right) \\
&= \frac{(2n-1)t}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} x^{2n-2} e^{-\frac{x^2}{2t}} dx \\
&= \ldots\ldots \\
&= \frac{(2n-1)!!t^n}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2t}} dx \\
&= \frac{(2n-1)!!t^n}{\sqrt{2\pi t}} \sqrt{2\pi t} \\
&= (2n-1)!!t^n
\end{aligned}$$

则可以得到：$E[B_t^2] = t, E[B_t^4] = 3t^2, E[B_t^6] = 15t^3, E[B_t^8] = 105t^4$ □