1. 指数鞅与布朗运动

Teorema 1

设 X 是连续过程，若

$$\forall \alpha \in \mathbb{R},\ M_t^\alpha = exp\left\{\alpha X_t - \frac{\alpha^2 t}{2}\right\}$$

是 F_t-鞅，则 X 是 F_t-布朗运动

proof. 由于 M_t^α 是 F_t-鞅，则对 $\forall\, 0 \leqslant t_0 \leqslant \cdots \leqslant t_n \leqslant t_{n+1}$ 有：

$$E\left[exp\left\{\alpha X_{t_{n+1}} - \frac{\alpha^2 t_{n+1}}{2}\right\} \mid X_{t_1}, X_{t_2}, \ldots, X_{t_n}\right] = exp\left\{\alpha X_{t_n} - \frac{\alpha^2 t_n}{2}\right\}$$

由于左式为：

$$\begin{aligned} & E\left[exp\left\{\alpha X_{t_{n+1}} - \frac{\alpha^2 t_{n+1}}{2}\right\} \mid X_{t_1}, X_{t_2}, \ldots, X_{t_n}\right] \\ = & E\left[exp\left\{\alpha(X_{t_{n+1}} - X_{t_n}) + \alpha X_{t_n} - \frac{\alpha^2 t_{n+1}}{2}\right\} \mid X_{t_1}, X_{t_2}, \ldots, X_{t_n}\right] \\ = & exp\left\{\alpha X_{t_n} - \frac{\alpha^2 t_{n+1}}{2}\right\} E\left[exp\left\{\alpha(X_{t_{n+1}} - X_{t_n})\right\}\right] \end{aligned}$$

不妨记 $B_{t_{n+1}-t_n} = X_{t_{n+1}} - X_{t_n}$，则有：

$$exp\left\{\alpha X_{t_n} - \frac{\alpha^2 t_{n+1}}{2}\right\} E\left[exp\left\{\alpha B_{t_{n+1}-t_n}\right\}\right] = exp\left\{\alpha X_{t_n} - \frac{\alpha^2 t_n}{2}\right\}$$

即：

$$E\left[exp\left\{\alpha B_{t_{n+1}-t_n}\right\}\right] = exp\left\{\frac{\alpha^2(t_{n+1} - t_n)}{2}\right\}$$

则由正态分布的矩母函数可知：从正态分布，即 $X_{t_{n+1}} - X_{t_n} \sim N(0, \sqrt{\alpha})$

下考虑独立增量性即可由于 $B_{t_{n+1}-t_n} \sim N(0, \sqrt{\alpha})$，则有：

$$E\left[B_{t_{n+1}-t_n} B_{t_{s+1}-t_s}\right] = 0$$

即

$$E\left[(X_{t_{n+1}} - X_{t_n})(X_{t_{s+1}} - X_{t_s})\right] = 0$$

综上，连续过程 X 是 F_t-布朗运动 □